THE STRUCTURE OF AUTOMORPHISM GROUPS OF SEMIGROUP INFLATIONS

GANNA KUDRYAVTSEVA

Department of Mechanics and Mathematics,
Kyiv Taras Shevchenko University,
Volodymyrs'ka, 60, Kuiv, 01033, Ukraine
akudr@univ.kiev.ua

Abstract

It is proved that the automorphism group of a semigroup being an inflation of its proper subsemigroup decomposes into a semidirect product of two groups one of which is a direct sum of full symmetric groups.

1 Introduction

In the study of a specific semigroup the description of all its automorphisms is one of the most important questions. The automorphism groups of many important specific semigroups are described (see, for example, the references in [1]). It happens that for two types of examples of semigroups having rather different nature the automorphism groups have similarities in their structure: each of them decomposes into a semidirect product of two groups one of which is a direct sum of the full symmetric groups. These two types of semigroups are variants of some semigroups of mappings (see [3], [2]) and maximal nilpotent subsemigroups of some transformation semigroups (see [4], [6]).

In the present paper we give a general setting for all these results via consideration of the automorphism group of an abstract semigroup being an inflation of its proper subsemigroup, constructed as follows. Define an equivalence h on a semigroup S via $(a,b) \in h$ if and only if $ax = bx$ and $xa = xb$ for all $x \in S$, i.e. a and b are not distinguished by multiplication

from either side. The notation h goes back to [3], where h was defined and studied for the variants of certain semigroups of mappings. Set further

$$\psi = (h \cap ((S \setminus S^2) \times (S \setminus S^2))) \cup \{(a, a) : a \in S^2\}.$$

Denote by T any transversal of ψ. Then T is a subsemigroup of S, and S is an inflation of T. An automorphism τ of T will be called *extendable* provided that τ coincides with the restriction to T of a certain automorphism of S. Clearly, all extendable automorphisms of T constitute a subgroup, H, of the group $AutT$ of all automorphisms of T.

Our main result is the following theorem.

Theorem 1. *The group* Aut$S$ *is isomorphic to a semidirect product of two groups one of which (the one which is normal) is the direct sum of the full symmetric groups on the* ψ*-classes and the other one is the group* H *consisting of all extendable automorphisms of* T*.*

In all the papers [3], [2], [4], [6] the semigroups under consideration are inflations on some their proper subsemigroups (although this is not mentioned explicitly in any of these papers), and one of the multiples (the one which is normal) in the structure theorem for automorphism group every time coincides with the corresponding multiple from Theorem 1 i.e. with the direct sum of the full symmetric groups on ψ-classes. Only the structure of the other multiple depends on the specific of a semigroup under consideration. However, in all the papers [3], [2], [4], [6] the structure of this multiple is easily understood, and the main points each time were to show that these easily understood automorphisms do not exhaust all the automorphisms of a given semigroup, to find the normal multiple from Theorem1 and to show that the construction of a semidirect product comes to the game.

2 Construction

A semigroup S is called an *inflation* of its subsemigroup (see [7], section 3.2) T provided that there is an onto map $\theta : S \to T$ such that:

- $\theta^2 = \theta$;
- $a\theta b\theta = ab$ for all $a, b \in S$.

In the described situation S is often referred to as an *inflation of* T *with an associated map* θ (or just *with a map* θ).

It is immediate that if S is an inflation of T then T is a retract of S (i.e. the image under an idempotent homomorphism) and that $S^2 \subset T$.

Lemma 1. *Suppose that S is an inflation of T with a map θ. Then* $\ker\theta \subset h$.

Proof. Let $(a, b) \in \ker\theta$ and $s \in S$. Then we have

$$as = a\theta s\theta = b\theta s\theta = bs;\ sa = s\theta a\theta = s\theta b\theta = sb.$$

It follows that $(a, b) \in h$. $\square$

Lemma 2. *The equivalence ψ, defined in the Introduction, is a congruence on S.*

Proof. Obviously, ψ is an equivalence relation. Prove that ψ is left and right compatible. Let $(a, b) \in \psi$ and $a \neq b$. Then $(a, b) \in (h \cap ((S \setminus S^2) \times (S \setminus S^2)))$. Let $c \in S$. As $(a, b) \in h$, one has $ac = bc$ and $ca = cb$ for each $c \in S$. It follows that $(ac, bc) \in \psi$ and $(ca, cb) \in \psi$ as ψ is reflexive. $\square$

Set T to be a transversal of ψ.

Lemma 3. *T is a subsemigroup of S, and S is an inflation of T.*

Proof. T is a subsemigroup of S as $T \supset S^2$. Let θ be the map $S \to T$ which sends any element x from S to the unique element of the ψ-class of x, belonging to T. The construction implies that S is an inflation of T with the map θ. $\square$

Let $S = \cup_{a \in T} X_a$ be the decomposition of S into the union of ψ-classes, where X_a denotes the ψ-class of a. Set G_a to be the full symmetric group acting on X_a and $G = \oplus_{a \in T} G_a$.

We start from the following easy observation

Lemma 4. *π is an automorphism of S for each $\pi \in G$.*

Proof. It is enough to show that $(xy)\pi = x\pi y\pi$ whenever $x, y \in S$. Suppose first that $x, y \in S \setminus S^2$. Since $xy \in S^2$ it follows that π stabilizes xy, so that $(xy)\pi = xy$. Now, the inclusions $(x, x\pi) \in h$ and $(y, y\pi) \in h$ imply $x\pi y\pi = x\pi y = xy$. This yields $xy\pi = x\pi y\pi$, and the proof is complete. $\square$

The following proposition gives a characterization of extendable automorphisms of T.

Proposition 1. *An automorphism τ of T is extendable if and only if the following condition holds:*

$$(\forall a, b \in T)\ \ a\tau = b \ \Rightarrow\ |X_a| = |X_b|. \tag{1}$$

Proof. Suppose $\tau \in \mathrm{Aut}T$ is extendable and $a \in T$. In the case when $a \in S \setminus S^2$ we have

$$X_a = \{b \in S \mid (a,b) \in h \text{ and } b \in S \setminus S^2\}.$$

Clearly, $(a,b) \in h \iff (a\tau, b\tau) \in h$ and $b \in S \setminus S^2 \iff b\tau \in S \setminus S^2$ for all $a, b \in S$. It follows that

$$X_{a\tau} = \{b\tau \mid b \in X_a\},$$

which implies Equation (1). The inclusion $a \in S^2$ is equivalent to $a\tau \in S^2$. But then $|X_a| = |X_{a\tau}| = 1$, which also implies Equation (1).

Suppose now that (1) holds for certain $\tau \in \mathrm{Aut}T$. Then one can extend τ to $\overline{\tau} \in AutS$ as follows.

Fix a collection of sets I_a, $a \in T$, and bijections $f_a : I_a \to X_a$, $a \in T$, satisfying the following conditions:

- $|I_a| = |X_a|$;
- $I_a = I_b$ whenever $|X_a| = |X_b|$;
- $I_a \cap I_b = \emptyset$ whenever $|X_a| \neq |X_b|$;
- if $a, b \in T$ and $|X_a| = |X_b|$ then $af_a^{-1} = bf_b^{-1}$.

Obviously, such collections I_a, $a \in T$, and f_a, $a \in T$, exist.

Consider $x \in S \setminus T$. Since T is a transversal of ψ there is $a \in T$ such that $x \in X_a$. By the hypothesis we have $|X_a| = |X_{a\tau}|$. Set $\overline{\tau}$ on X_a to be the map from X_a to $X_{a\tau}$, defined via $x \mapsto xf_a^{-1}f_{a\tau}$. In this way we define a bijection $\overline{\tau}$ of S such that $\overline{\tau}|_T = \tau$. It will be called an *extension* of τ to S. To complete the proof, we are left to show that $\overline{\tau}$ is a homomorphism. Let $x, y \in S$, $x \in X_a$, $y \in X_b$. Then

$$(xy)\overline{\tau} = (ab)\tau = a\tau b\tau = x\overline{\tau}y\overline{\tau},$$

as required.

□

Let $\tau \in H$. Of course, $\overline{\tau}$, constructed in the proof of Proposition 1, depends not only on τ, but also on the sets I_a and the maps f_a, so that τ may have several extentions to S. Fix any extention $\overline{\tau}$ of τ.

Lemma 5. *$\tau \mapsto \overline{\tau}$ is an embedding of H into* Aut$S$.

Proof. Proof follows directly from the construction of $\overline{\tau}$. □

Denote by $\overline{H}$ the image of H under the embedding of H into Aut$S$ from Lemma 5.

3 Proof of Theorem 1

Proposition 2. *$\overline{H}$ acts on G by automorphisms via $\pi^\tau = \tau^{-1}\pi\tau$, $\tau \in \overline{H}, \pi \in G$.*

Proof. Let $\pi \in G$ and $\tau \in \overline{H}$. Show first that $\pi^\tau \in G$. Take any $x \in S$. Let X_a be the block which contains x. We consequently have that $x\tau^{-1} \in X_{a\tau^{-1}}$, $x\tau^{-1}\pi \in X_{a\tau^{-1}}$ and $x\tau^{-1}\pi\tau \in X_{a\tau^{-1}\tau} = X_a$. Hence, $x\pi^\tau \in X_a$. It follows that $\pi^\tau \in G$.

That $\pi \mapsto \pi^\tau$ is one-to-one, onto and homomorphic immediately follows from the definition of this map. We are left to show that the map, which sends $\tau \in \overline{H}$ to $\pi \mapsto \pi^\tau \in \mathrm{Aut}G$, is homomorphic. Indeed, $\pi^{\tau_1\tau_2} = (\tau_1\tau_2)^{-1}\pi(\tau_1\tau_2) = (\pi^{\tau_1})^{\tau_2}$. This completes the proof. □

In the following two lemmas we show that G and $\overline{H}$ intersect by the identity automorphism and generate $\mathrm{Aut}S$.

Lemma 6. *$G \cap \overline{H} = id$, where id is the identity automorphism of S.*

Proof. The proof follows from the observation that the decomposition $S = \cup_{a\in T} X_a$ is fixed by each element of G, while only by the identity element of $\overline{H}$. □

Lemma 7. $\mathrm{Aut}S = \overline{H} \cdot G$.

Proof. Let $\varphi \in \mathrm{Aut}S$. It follows from the definition of ψ that φ maps each ψ-class onto some other ψ-class. Define a bijection $\tau : T \to T$ via $a\tau = b$, if $X_a\varphi = X_b$ and show that τ is an extendable automorphism of T. It follows immediately from the construction of τ that Equation (1) holds, so that τ is extendable by Proposition 1. Let $\overline{\tau} \in \mathrm{Aut}S$ be an extension of τ. The construction implies that $\varphi(\overline{\tau})^{-1} \in G$. □

Now the proof of Theorem 1 follows from Proposition 2 and Lemmas 6 and 7.

References

[1] F. Szechtman *On the automorphism group of the centralizer of an idempotent in the full transformation monoid.* Semigroup Forum, 70 (2005), p.238-242.

[2] G. Kudryavtseva and G. Tsyaputa. *The automorphism group of the sandwich inverse symmetric semigroup.* Visnyk of Kyiv University, 13-14 (2005), p. 101-105. (in Ukrainian)

[3] J.S.V. Symons *On a generalization of the transformation semigroup.* J. Austral Math. Soc., 19 (1975), p. 47-61.

[4] O. Ganyushkin, S. Temnikov and H. Shafranova *Automorphism groups of maximal nilpotent subsemigroups of the semigroup $\mathcal{IS}M$.* Math. Stud., 13 (2000), 1, p. 11-22.

[5] P.M. Higgins. *Techniques of semigroup theory.* Oxford University Press, 1992.

[6] A. Stronska. *On the automorphisms for the nilpotent subsemigroups of the order-decreasing transformations semigroup.* to appear in Quest. Alg.

[7] A.H. Clifford, G.B. Preston, *The algebraic theory of semigroups*, vol. 1,2, AMS, 1964, 1967.